LOWER BOUNDS FOR h–VECTORS OF k–CM, INDEPENDENCE AND BROKEN CIRCUIT COMPLEXES

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Abstract. We present a number of lower bounds for the h–vectors of k–CM, broken circuit and independence complexes. These lead to bounds on the coefficients of the characteristic and reliability polynomials of matroids. The main techniques are the use of series and parallel constructions on matroids and the short simplicial h–vector for pure complexes.

1. Introduction

Based on the ideas of Whitney [27] and Rota [21], the broken circuit complex of a graph was introduced by Wilf in "What polynomials are chromatic?" [29] Extended to matroids by Brylawski [9], its f–vector corresponds to the coefficients of the characteristic polynomial of the matroid. The h–vector encodes the same information in a different way. From the point of view of matroids, Wilf's original question becomes, "What are the possible f–vectors, or equivalently h–vectors, of broken circuit complexes of matroids?"

Cohen–Macaulay complexes cover a wide variety of examples. In addition to the broken circuit and independence complexes of matroids covered here, Cohen–Macaulay complexes also include all triangulations of homology balls and spheres. In contrast to broken circuit complexes, the possible h–vectors (and hence f–vectors) of Cohen–Macaulay complexes have been completely characterized (see, for instance, [24, Theorem II.3.3, pg. 59]). Introduced by Baclawski, doubly Cohen–Macualay complexes are Cohen–Macaulay complexes which neither lose a dimension nor lose the Cohen–Macaulay property when any vertex is removed. Spheres are doubly Cohen–Macaulay but balls are not. More generally, a Cohen–Macaulay complex is k–CM if it retains its dimension and is still Cohen–Macaulay whenever $k-1$ or fewer vertices are removed. In addition to the independence complexes considered below, the order complex of a geometric lattice with the top and bottom points removed is k–CM if every line has at least k points [2].

The h–vectors of independence complexes of matroids are contained in the intersection of h–vectors of broken circuit complexes and k–CM complexes. Precisely, the cone on any independence complex is a broken circuit complex. In addition, if the smallest cocircuit of the matroid has cardinality k, then its independence complex is a k–CM complex. The close connection between h–vectors of independence

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complexes of matroids and reliability problems has been studied by a number of authors. See [13] for a recent survey.

Upper bounds on all of the above complexes have been studied. As they are all Cohen–Macaulay they share a common absolute upper bound of $h_i \le \binom{n-r-1+i}{i}$, where n is the number of vertices and $(r-1)$ is the dimension of the complex. In addition, they all satisfy the relative upper bound $h_{i+1} \le h_i^{<i>}$ (see Section 4 for a definition of $h_i^{<i>}$).

Our main purpose is to analyze absolute and relative lower bounds for the h–vectors of k–CM, broken circuit and independence complexes. Section 2 contains the basic facts of the short–simplicial h–vector. The main tool for providing relative lower bounds is equation (6). The broken circuit and independence complex of a matroid are described in section 3. Sections 4, 5 and 6 contain absolute and relative lower bounds for k–CM, broken circuit and independence complexes respectively.

Throughout the paper Δ is an $(r-1)$–dimensional simplicial complex with vertex set $V, |V| = n$. The link of a vertex $v \in V$ is $lk_\Delta v$, or just $lk\ v$ if no confusion is possible. We use $\Delta - v$ for the complex obtained by removing v and all of the faces which contain v from Δ. Similarly, if $A \subseteq V$, then, $\Delta - A$ is the complex obtained by removing all of the vertices in A and any faces which contain one or more of those vertices.

2. Face enumeration

The combinatorics of a simplicial complex Δ can be encoded in several ways. The most direct is to let $f_i(\Delta)$ be the number of faces of cardinality i. For an $(r-1)$–dimensional complex the h–vector of Δ is the sequence $(h_0(\Delta), \dots, h_r(\Delta))$, where

$$h_i(\Delta) = \sum_{j=0}^{i} (-1)^{i-j} \binom{r-j}{r-i} f_j(\Delta). \tag{1}$$

Equivalently,

$$f_j(\Delta) = \sum_{i=0}^{j} \binom{r-i}{r-j} h_i(\Delta). \tag{2}$$

By convention, $h_i(\Delta) = f_i(\Delta) = 0$ if $i < 0$ or $i > r$. The *short simplicial* h–vector was introduced in [16] as a simplicial analogue of the short cubical h–vector in [1]. It is the sum of the h–vectors of the links of the vertices. As far as we know, (5) was first stated in [17]. However, only a proof for shellable Δ was given there. So, we include a proof for arbitrary pure complexes for the sake of completeness.

Definition 2.1. *Let Δ be a pure simplicial complex . Define*

$$\tilde{h}_i(\Delta) = \sum_{v \in V} h_i(lk\ v). \tag{3}$$

Lemma 2.2. [16] *Let Δ be a pure simplicial complex. For all $i, 0 \le i \le r-1$,*

$$\tilde{h}_i(\Delta) = \sum_{j=0}^{i} (-1)^{i-j} (j+1) \binom{r-j-1}{r-i-1} f_{j+1}. \tag{4}$$

Proposition 2.3. *Let Δ be a pure simplicial complex. Then,*

$$\tilde{h}_{i-1}(\Delta) = i\ h_i(\Delta) + (r-i+1)h_{i-1}(\Delta). \tag{5}$$

If $\dim(\Delta - v) = r-1$ *for every vertex* v*, then*

$$\sum_{v \in V} h_i(\Delta - v) = (n-i)h_i(\Delta) - (r-i+1)h_{i-1}(\Delta). \tag{6}$$

Proof. Combining (2) and (4),

$$\begin{aligned}
\tilde{h}_{i-1}(\Delta) &= \sum_{j=0}^{i-1}(-1)^{i-j-1}(j+1)\binom{r-j-1}{r-i}\sum_{k=0}^{j+1}\binom{r-k}{r-j-1}h_k(\Delta) \\
&= \sum_{k=0}^{i} h_k(\Delta)\left\{\sum_{j=k-1}^{i-1}(-1)^{i-j-1}(j+1)\binom{r-j-1}{r-i}\binom{r-k}{r-j-1}\right\} \\
&= \sum_{k=0}^{i} h_k(\Delta)\left\{\sum_{j=k-1}^{i-1}(-1)^{i-j-1}(j+1)\binom{r-j-1}{i-j-1}\binom{r-k}{j+1-k}\right\}.
\end{aligned}$$

Substituting $s = j-k+1$ and $t = i-j-1$,

$$\begin{aligned}
\tilde{h}_{i-1}(\Delta) &= \sum_{k=0}^{i} h_k(\Delta)\left\{\sum_{s+t=i-k}(-1)^t(i-t)\binom{r+t-i}{t}\binom{r+s+t-i}{s}\right\} \\
&= \sum_{k=0}^{i} h_k(\Delta)\left\{\sum_{s+t=i-k}(-1)^t(i-t)\frac{A}{s!t!}\right\},
\end{aligned}$$

where A is the falling factorial $(r-k)\cdot(r-k-1)\cdots(r-i+1)$.

For a fixed i, define c_k by

$$c_k = \sum_{s+t=i-k}(-1)^t(i-t)\frac{1}{s!t!}.$$

Equation (5) is equivalent to showing that $c_i = i, c_{i-1} = 1$ and $c_k = 0$ in all other cases. This can be seen by recognizing c_{i-k} as the k^{th} term in the generating series for

$$(i+x)e^{-x}\cdot e^x = \left(\sum_{t=0}^{\infty}(-1)^t\frac{(i-t)}{t!}x^t\right)\left(\sum_{s=0}^{\infty}\frac{1}{s!}x^s\right).$$

In order to prove that (6) holds, we first notice that the hypothesis implies that $h_i(\Delta) = h_i(\Delta - v) + h_{i-1}(lk\ v)$ for every vertex v. Now sum this equation over all the vertices and apply equation (5). □

The above proposition makes precise the idea that, taken together, $h_{i-1}(\Delta)$ and $h_i(\Delta)$ measure the "average contribution of $h_{i-1}(lk\ v)$ to $h_i(\Delta)$." Another consequence of (5) is that if the automorphism group of a pure $(r-1)$–dimensional complex Δ is transitive, or more generally if $h_{i-1}(lk\ v)$ is independent of v, then n divides $\{i\ h_i(\Delta) + (r-i+1)h_{i-1}(\Delta)\}$.

3. Broken circuit and independence complexes of matroids

We follow [19] for matroid terminology. Unless otherwise specified, M is always a rank r matroid with ground set E (or $E(M)$ if necessary) and $|E| = n$. There are many equivalent ways of defining matroids. The most convenient for us is the following.

A *matroid*, M, is a pair $(E, \mathcal{I}), E$ a non-empty finite ground set and $\mathcal{I}$ a distinguished set of subsets of E. The members of $\mathcal{I}$ are called the *independent* subsets of M and are required to satisfy:

(1) The empty set is in $\mathcal{I}$.
(2) If B is an independent set and $A \subseteq B$, then A is an independent set.
(3) If A and B are independent sets such that $|A| < |B|$, then there exists an element $x \in B - A$ such that $A \cup x$ is independent.

Matroid theory was introduced by Whitney [28]. The prototypical example of a matroid is a finite subset of a vector space with the canonical independent sets. Another example is the cycle matroid of a graph. Here the ground set is the edge set of the graph and a collection of edges is independent if and only if it is acyclic.

An element e of a matroid is a *loop* if it is not contained in any independent set. The *circuits* of a matroid are its minimal dependent sets. Every loop of M is a circuit. A maximal independent set is called a *basis*, and any element which is contained in every basis is a *coloop* of the matroid. Every basis of M has the same cardinality. The *rank* of M, or $r(M)$, is that common cardinality. Similarly, the rank of a subset A of E is the cardinality of any maximal independent subset of A and is denoted $r(A)$. The *deletion* of M at e is denoted $M - e$. It is the matroid whose finite set is $E - e$ and whose independent sets are simply those members of $\mathcal{I}$ which do not contain e. The *contraction* of M at e is denoted M/e. It is a matroid whose ground set is also $E - e$. If e is a loop or a coloop of M then $M/e = M - e$. Otherwise, a subset I of $E - e$ is independent in M/e if and only if $I \cup e$ is independent in M. Deletion and contraction for a subset A of E is defined by repeatedly deleting or contracting each element of A.

The *dual* of M is $M^\star$. It is the matroid whose ground set is the same as M and whose bases are the complements of the bases of M. For example, $U_{i,j}$ is the matroid defined by $E = \{1, 2, \ldots, j\}$ and $\mathcal{I} = \{A \subseteq E : |A| \leq i.\}$ So, $U^\star_{i,j} = U_{j-i,j}$.

Two non-loop elements $e, f \in E$ are *parallel* if they form a circuit. The relation "is parallel to" is an equivalence relation on E and the corresponding equivalence classes are the parallel classes of M. If P is a parallel class of M, then for any $e \in P$ all of the members of $P - e$ are loops in M/e. A parallel class in $M^\star$ is a *series class* of M. If S is a series class of M, then for any $e \in S$, all of the members of $S - e$ are coloops in $M - e$.

Let $M = (E, \mathcal{I})$ and $M' = (E', \mathcal{I}')$ be two matroids with $E \cap E' = \emptyset$. Then $M \oplus M'$ is the direct sum of M and M'. It is the matroid whose ground set is $E \cup E'$ and whose independent sets are those subsets of the form $I \cup I', I \in \mathcal{I}, I' \in \mathcal{I}'$. A matroid is *connected* if it is not the direct sum of two smaller matroids. Every matroid can be written uniquely (up to order) as a direct sum $M = M_1 \oplus \cdots \oplus M_k$ of connected matroids. The *components* of M are the summands of this decomposition.

The *independence complex* of M is

$$\Delta(M) = \{A \subseteq E : A \text{ is independent}\}.$$

Evidently, $\Delta(M)$ is a pure $(r-1)$–dimensional complex, where r is the rank of M. In addition, $\Delta(M-e) = \Delta(M)-e$ and if e is not a loop of M, then $\Delta(M/e) = lk_{\Delta(M)}\, e$.

In order to define the broken circuit complex for M, we first choose a linear order ω on the elements of the matroid. Given such an order, a *broken circuit* is a circuit with its least element removed. The *broken circuit complex* is the simplicial complex whose simplices are the subsets of E which do not contain a broken circuit. We denote the broken circuit complex of M and ω by $\Delta^{BC}(M)$, or $\Delta^{BC}(M,\omega)$. Different orderings may lead to different complexes, see [3, Example 7.4.4]. However, $f_i(\Delta^{BC}(M,\omega))$ does not depend on ω (see Theorem 3.2 below). Conversely, distinct matroids can have the same broken circuit complex. For instance, let $E = \{e_1, e_2, e_3, e_4, e_5, e_6\}$, and let ω be the obvious order. Let M_1 be the matroid on E whose bases are all triples except $\{e_1, e_2, e_3\}$ and $\{e_4, e_5, e_6\}$ and let M_2 be the matroid on E whose bases are all triples except $\{e_1, e_2, e_3\}$ and $\{e_1, e_5, e_6\}$. Then M_1 and M_2 are non–isomorphic matroids but their broken circuit complexes are identical.

In order to easily distinguish the h–vectors of $\Delta(M)$ and $\Delta^{BC}(M)$ we use the following notation.

Definition 3.1. *Let M be a rank r matroid.*

- $h_i(M) = h_i(\Delta(M))$.
- $b_i(M) = h_{r-i}(\Delta^{BC}(M))$.
- $w_i(M) = f_{r-i}(\Delta^{BC}(M))$.
- $b^\star_i(M) = b_i(M^\star) = h_{n-r-i}(\Delta^{BC}(M^\star))$

We will suppress the M when there is no danger of confusion. The invariants $h_i, b_i, w_i, b^\star_i$ are closely related to the Tutte polynomial of M. The *Tutte polynomial* is a two–variable polynomial invariant of M defined by

$$T(M;x,y) = \sum_{A\subseteq E} (x-1)^{r(M)-r(A)}(y-1)^{|A|-r(A)}.$$

Theorem 3.2. [3] *Suppose M has k components and j coloops. Then,*

a. $T(M;x,1) = h_0x^r + h_1x^{r-1} + \cdots + h_{r-j}x^j$.
b. $T(M;x,0) = b_rx^r + b_{r-1}x^{r-1} + \cdots + b_kx^k$.
c. $T(M;0,y) = b^\star_{n-r}x^{n-r} + \cdots + b^\star_kx^k$.
d. $(-1)^rT(M;1-x,0) = w_0x^r - w_1x^{r-1} + \cdots + (-1)^rw_r$.

The w_i are the unsigned Whitney numbers of the first kind. The *characteristic polynomial* of M is $(-1)^rT(M;1-x,0)$. The characteristic polynomial of a matroid has a number of applications including graph coloring and flows, linear coding theory and hyperplane arrangements. See [12] for a survey.

Properties [a]-[d] of b_i and h_i listed below follow immediately from corresponding properties of the Tutte polynomial which can be found in [11]. The parallel and series connection of two (pointed) matroids is described in [19, Section 7.1].

Theorem 3.3 (Tutte recursion)**.**

a. *If M has j coloops, then $h_i(M) = h_i(\tilde{M})$, where $\tilde{M}$ is M with the coloops deleted. In particular, $h_i(M) > 0$ if and only if $0 \le i \le r-j$.*
b. *If M has k components and no loops, then $b_i > 0$ if and only if $k \le i \le r$.*
c. *If e is neither a loop nor a coloop of M, then $h_i(M) = h_i(M-e)+h_{i-1}(M/e)$ and $b_i(M) = b_i(M-e) + b_i(M/e)$.*

d. *If* $M = M_1 \oplus M_2$*, then* $h_i(M) = \sum_{j+k=i} h_j(M_1)h_k(M_2)$ *and* $b_i(M) = \sum_{j+k=i} b_j(M_1)b_k(M_2)$.

e. *Suppose that* P *is a parallel class of* M*. Let* $\tilde{M}$ *be* M *with all but one element, say* e*, of* P *deleted. Then,* $h_i(M) = h_i(\tilde{M}) + (|P|-1)h_{i-1}(\tilde{M}/e)$.

f. *Let* S *be a series class of* M*. Let* $\tilde{M}$ *be* M *with all but one element, say* e*, of* S *contracted. Then* $b_i(M) = b_i(\tilde{M}) + \sum_{j=1}^{|S|-1} b_{i-j}(\tilde{M} - e)$.

g. *Let* M *be a parallel connection of* A *and* B*, where the rank of* A *is* $r(A)$ *and the rank of* B *is* $r(B)$*. The rank of* M *is* $r(A)+r(B)-1$*. In addition,* $b_i(M) = \sum_{j+k=i+1} b_j(A)b_k(B)$*. If* A *and* B *are connected, then* M *is also connected.*

Proof. Property [g] follows from the fact that if M is a parallel connection of A and B, then $T(M;x,0) = T(A;x,0) * T(B;x,0)/x$ [11, pg. 179–182]. Both [e] and [f] are proved by deleting and contracting all the elements of the given parallel or series class except e. □

One of the consequences of [a] and [f] above is that if we increase the size of a series class of cardinality k in M by one, then $b_1, \dots, b_k$ are unchanged, while b_i for $i > k$ may increase.

4. Cohen–Macaulay and k–CM complexes

There are several equivalent definitions of Cohen–Macaulay complexes. The following will suffice for our purposes.

Definition 4.1. *A pure* $(r-1)$*–dimensional complex* Δ *is Cohen–Macaulay if for every face* $F \in \Delta$ *and* $i < \dim(lk\ F), \tilde{H}_i(lk\ F;\mathbb{Q}) = 0$.

A numerical description of all possible h–vectors of Cohen–Macaulay complexes can be given using the following operator. Given any positive integers h and i there is a unique way of writing

$$h = \binom{a_i}{i} + \binom{a_{i-1}}{i-1} + \cdots + \binom{a_j}{j}$$

so that $a_i > a_{i-1} > \cdots > a_j \geq j \geq 1$. Define

$$h^{<i>} = \binom{a_i+1}{i+1} + \binom{a_{i-1}+1}{i} + \cdots + \binom{a_j+1}{j+1}$$

Theorem 4.2. [24] *A sequence of non–negative integers* $(h_0, \dots, h_r)$ *is the* h*–vector of some Cohen–Macaulay complex if and only if* $h_0 = 1$ *and* $h_{i+1} \leq h_i^{<i>}$ *for all* $1 \leq i \leq r-1$.

The notion of k–CM complexes was introduced by Baclawski [2].

Definition 4.3. *Let* Δ *be a pure* $(r-1)$*–dimensional simplicial complex with vertex set* V *and* $k \geq 1$*. We say that* Δ *is* k*–CM if for all* $A \subseteq V$ *with* $|A| < k, \Delta - A$ *is Cohen–Macaulay of dimension* $(r-1)$.

Examples of 2–CM complexes include order complexes of geometric lattices, finite buildings and triangulations of spheres. Several examples and constructions involving k–CM complexes, especially for order complexes of posets, are contained in [2]. Since $lk_\Delta v - A = lk_{\Delta - A} v$, the link of any vertex of a k–CM complex is k–CM, and removing a vertex from a k–CM complex leaves a $(k-1)$–CM complex (as long as $k > 1$).

The independence and broken circuit complexes of a matroid are Cohen–Macaulay [23]. So, $\Delta(M)$ is k–CM if and only if every hyperplane of M has cardinality at most $n-k$. Equivalently, the smallest cocircuit of M has at least k elements. However, $\Delta^{BC}(M)$ is a cone on the least element, hence it is only 1–CM. If the cone point is removed, then the remaining complex is also Cohen–Macaulay, but may still be only 1–CM. For example, let M be the cycle matroid of the theta–graph with three paths each of length 2. Direct computation shows that the h–vector of $\Delta^{BC}(M)$ is $(1,2,3,1)$. Removing the cone point leaves a 2–dimensional complex with 5 points and the same h–vector. By Corollary 4.5 below, $(1,2,3,1)$ is not the h–vector of any 2–dimensional 2–CM complex with 5 points.

Theorem 4.2 gives an upper bound for possible h–vectors of Cohen–Macaulay complexes. It also makes it clear that there are no lower bounds. For k–CM complexes we have the following absolute lower bound. Recall that $U_{r,n}$ is the rank r matroid with n elements such that every r–element subset is a basis.

Proposition 4.4. *Let Δ be an $(r-1)$–dimensional k–CM complex. Then,*

$$h_i(\Delta) \geq h_i(U_{r,r+k-1}).$$

Proof. Induction on n and k. When $k = 1$, the theorem is simply the statement that $h_i(\Delta) \geq 0$ for $i \geq 1$, and $h_0(\Delta) \geq 1$. For fixed k, the definition of k–CM forces $n \geq r + k - 1$. Suppose $n = r + k - 1$. Since the removal of any subset of vertices of cardinality $k-1$ does not lower the dimension of Δ, every subset of vertices of cardinality r must be a face of Δ. So, $\Delta = \Delta(U_{r,r+k-1})$. For the induction step, let v be any vertex of Δ. Then

$$h_i(\Delta) = h_i(\Delta - v) + h_{i-1}(lk_\Delta v) \geq h_i(U_{r,r+k-2}) + h_{i-1}(U_{r-1,r+k-2}) = h_i(U_{r,r+k-1}).$$

□

Minimizing h–vectors is closely related to the problem of finding the least reliable graph. Let G be a connected graph with $r+1$ vertices and n edges. Thus $M(G)$, the cycle matroid of G, has rank r and cardinality n. Suppose that each edge of G has equiprobability $p, 0 < p < 1$ of being deleted. Then the probability that G remains connected is $R_G(p) = (1-p)^r[h_0(M(G)^\star) + h_1(M(G)^\star)p + \cdots + h_{n-r}(M(G)^\star)p^{n-r}]$. Boesch, Satyanarayana and Suffel posed the problem of finding the minimum of $R_G(p)$ among all connected simple graphs with $r+1$ vertices and n edges. They also conjectured that a particular graph, which they called $L(r+1,n)$, would attain that lower bound [4]. Brown, Colbourn and Devitt further conjectured that the h–vector of $L(r+1,n)$ would be an absolute lower bound for the h–vector of $M(G)^\star$ among all connected simple graphs with $r+1$ vertices and n edges [7]. The original conjecture of Boesch et. al. was confirmed for n greater than $\binom{r-1}{2}$ in [20]. The corresponding problem in the category of matroids is to find among all rank r cosimple matroids of cardinality n one which minimizes the h–vector. Since M is cosimple if and only if $\Delta(M)$ is 3–CM, the above proposition shows that $U_{0,n-r-2} \oplus U_{r,r+2}$ is the solution to this problem.

Combining the above proposition with (6) immediately gives a relative lower bound.

Corollary 4.5. *Let Δ be an $(r-1)$–dimensional k–CM complex with n vertices. Then,*

$$(n-i)h_i \geq (r-i+1)h_{i-1} + n\binom{i+k-3}{i}.$$

Proof. For every vertex $v, \Delta - v$ is $(k-1)$–CM. Now combine (6), Proposition 4.4 and the fact that $h_i(U_{r,r+k-2}) = \binom{i+k-3}{i}$. □

Problem 4.6. *Given r, n, k and i, what is the minimum of $h_i(\Delta)$ over all $(r-1)$–dimensional k–CM complexes with n vertices? Does there exist a Δ which attains these values?*

Conjecture II.6.2 in [24] would imply that for 2–CM complexes with n equal to $r+2, h_i(\Delta) \geq h_i(\Delta(U_{1,2} \oplus U_{r-1,r}))$. In section 6 we will give an answer to this problem for independence complexes of matroids when n is sufficiently large.

5. Broken circuit complexes

In this section we assume that M has no loops. An absolute upper bound for b_i when $1 \leq i \leq r$ is $\binom{n-i-1}{r-i}$ and this is achieved by $U_{n,r}$. Theorem 4.2 gives a relative upper bound of $b_{r-i} \leq b_{r-i+1}^{<i-1>}$. Absolute lower bounds for b_i were determined by Brylawski.

Theorem 5.1. [10] *If M is as above, then $b_i \geq n-r$ for all $i, 2 \leq i \leq r-1$.*

In order to find relative lower bounds for b_1 we introduce the following definition.

Definition 5.2. *Let S be a series class of a connected matroid M. Then S is a* **regular series** *class of M if $M-S$ is connected.*

Proposition 5.3. *If M is connected and contains more than one series class, then M contains at least three regular series classes.*

Proof. Induction on m, the number of series classes in M. A matroid with exactly two series class is not connected. If $m=3$, then M is the cycle matroid of a theta graph with exactly three paths. In this case all three of the series classes are regular.

For the induction step, let S be a series class which is not regular. Let $\tilde{M}$ be the matroid obtained by contracting all but one of the elements of S. Let e be the remaining element of S. Since $\tilde{M}$ is connected, but $\tilde{M}-e$ is not connected, $\tilde{M}$ is the series connection of two connected matroids A and B at e [19, Theorem 7.1.16]. Both A and B must contain more than one series class, otherwise they would be contained in S. Therefore, the induction hypothesis applies to A and B. Even if $\{e\}$ is contained in a regular series class in A and B, both A and B contain two other regular series classes. All four of these series classes are regular in M. □

Theorem 5.4. *If M is connected and $1 \leq i \leq r$, then*

$$b_i \leq \binom{r-2}{i-1}b_1 + \binom{r-2}{i-2}. \tag{7}$$

Proof. The proof is by induction on n, the initial case being the three–point line. Let S be a series class of M. If S is the only series class of M, then M is a circuit and (7) holds. Otherwise, by the previous proposition, we may choose S to be a regular series class. In particular, $M-S$ is connected. We break the induction step into three cases.

(1) $M-S$ and M/S are connected: Let $s=|S|$. If $s>i$, then $b_i(M)=b_i(\hat{M})$ and $b_1(M)=b_1(\hat{M})$, where $\hat{M}$ is M with S contracted down to a series class of cardinality i. So, we will assume that $s\le i$. Let $\tilde{M}$ be M with S contracted down to a single element e. Since M is connected, e is neither a loop nor a coloop of M. Applying Tutte recursion to M and then again to $\tilde{M}$ we see that

$$b_i(M)=b_i(\tilde{M}/e)+\sum_{j=0}^{s-1} b_{i-j}(\tilde{M}-e).$$

Now, since $\tilde{M}/e=M/S$ is a rank $r-s$ connected matroid and $\tilde{M}-e=M-S$ is a rank $r-s+1$ connected matroid, the induction hypothesis implies that the above expression is bounded above by

$$\binom{r-s-2}{i-1}b_1(\tilde{M}/e)+\binom{r-s-1}{i-2}+\sum_{j=0}^{s-1}\binom{r-s-1}{i-j-1}b_1(\tilde{M}-e)+\sum_{j=0}^{s-1}\binom{r-s-1}{i-j-2}$$

$$\le\binom{r-2}{i-1}b_1(\tilde{M}/e)+\binom{r-2}{i-1}b_1(\tilde{M}-e)+\binom{r-2}{i-2}+$$

$$\left\{\binom{r-s-2}{i-1}-\binom{r-2}{i-1}\right\}b_1(\tilde{M}/e)+\binom{r-s-2}{i-2}.$$

Since $\tilde{M}/e$ is connected, $b_1(\tilde{M}/e)\ge 1$. Thus, the last row is non–positive and (7) is satisfied. To see the last inequality, note that

$$\sum_{j=0}^{s-1}\binom{r-s-1}{i-j-1}\le\sum_{j=0}^{s-1}\binom{r-s-1}{i-j-1}\binom{s-1}{j}=\binom{r-2}{i-1},$$

and similarly,

$$\sum_{j=0}^{s-1}\binom{r-s-1}{i-j-2}\le\sum_{j=0}^{s-1}\binom{r-s-1}{i-j-2}\binom{s-1}{j}=\binom{r-2}{i-2}.$$

(2) $S=\{e\}, M-e$ is connected, but M/e is not connected: Then, M is the parallel connection of two connected matroids A and B with $r(A)+r(B)-1=r$ [19, Theorem 7.1.16]. By Theorem 3.3 and the induction hypothesis,

$$b_i(M)=\sum_{j+k-1=i} b_j(A)b_k(B)$$

$$
\begin{aligned}
&\le \sum_{j+k-1=i} \left\{ \binom{r(A)-2}{j-1} b_1(A) + \binom{r(A)-2}{j-2} \right\} \left\{ \binom{r(B)-2}{k-1} b_1(B) + \binom{r(B)-2}{k-2} \right\} \\
&= \sum_{j+k-1=i} \left\{ \binom{r(A)-2}{j-1}\binom{r(B)-2}{k-1} b_1(A)b_1(B) + \binom{r(A)-2}{j-1}\binom{r(B)-2}{k-2} b_1(A) \right\} + \\
&\quad \sum_{j+k-1=i} \left\{ \binom{r(A)-2}{j-2}\binom{r(B)-2}{k-1} b_1(B) + \binom{r(A)-2}{j-2}\binom{r(B)-2}{k-2} \right\} \\
&= \binom{r-3}{i-1} b_1(A)b_1(B) + \binom{r-3}{i-2} b_1(A) + \binom{r-3}{i-2} b_1(B) + \binom{r-3}{i-3}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\binom{r-2}{i-1} b_1(M) + \binom{r-2}{i-2} - b_i(M) \\
\ge\ & \left\{ \binom{r-2}{i-1} - \binom{r-3}{i-1} \right\} b_1(A)b_1(B) + \left\{ \binom{r-2}{i-2} - \binom{r-3}{i-3} \right\} - \\
&\binom{r-3}{i-2} \{ b_1(A) + b_1(B) \} \\
=\ & \binom{r-3}{i-2} \left(b_1(A)b_1(B) + 1 - b_1(A) - b_1(B) \right) \ge 0.
\end{aligned}
$$

(3) Finally, suppose that S is a non–trivial series, $M - S$ is connected, but M/S is not connected. Let $s, \tilde{M}$ and e be as above. Since $\tilde{M}/e$ is not connected, $b_1(\tilde{M}) = b_1(\tilde{M} - e)$. Therefore,

$$
\begin{aligned}
b_i(M) &= b_i(\tilde{M}) + \sum_{j=1}^{s-1} b_{i-j}(\tilde{M} - e) \\
&\le b_1(M) \left\{ \sum_{j=0}^{s-1} \binom{r-s-1}{i-j-1} \right\} + \sum_{j=0}^{s-1} \binom{r-s-1}{i-j-2} \\
&\le \binom{r-2}{i-1} b_1(M) + \binom{r-2}{i-2}.
\end{aligned}
$$

□

Corollary 5.5. *Let M be a rank r matroid with k components, $r - k \ge 2$. Let $2 \le i \le r - k$. Then,*

$$
b_{i+k-1}(M) \le \binom{r-k-1}{i-1} b_k(M) + \binom{r-k-1}{i-2}. \tag{8}
$$

Proof. Since $k = 1$ is the previous theorem we assume that M is not connected. Let $M = M_1 \oplus \cdots \oplus M_k$ be a direct sum decomposition of M into connected matroids. Define $\tilde{M}_1 = M_1$. Given $\tilde{M}_i$ let $\tilde{M}_{i+1}$ be any parallel connection of $\tilde{M}_i$ and M_{i+1}. Then $\tilde{M}_k$ is a connected matroid of rank $r - k + 1$. Furthermore, by Theorem 3.3 $b_{i+k-1}(M) = b_i(\tilde{M}_k)$. Since (7) holds for the connected $\tilde{M}$, (8) holds for M. □

When does equality occur in the above theorem? The proof shows that if equality occurs, then it must also occur in the minors of M used in the induction. Combining this with an induction argument shows that if $b_i(M) = \binom{r-2}{i-1}b_1(M) + \binom{r-2}{i-2}$, then $b_j(M) = \binom{r-2}{j-1}b_1(M)+\binom{r-2}{j-2}$ for all $1 \le j \le i$. Brylawski proved (7) for $i = r-1$. He also showed that given b_1 and r, then equality occurs if M is the parallel connection of a (b_1+2)–point line and $r-1$ three–point lines. Hence, (7) is optimal, although a complete description of the matroids which satisfy equality in this corollary remains unknown [10].

The coefficient $b_1(M)$ is also known as $\beta(M)$, the *beta* invariant of M. Brylawski identified matroids with beta invariant 1 as series–parallel matroids [8] while Oxley classified matroids with $2 \le \beta(M) \le 4$ [18].

Theorem 5.6. *Assume $r \ge 2$ and let $\beta = b_1(M)$. Then, for all $i, 0 \le i \le r$,*

$$w_i \le \sum_{j=0}^{i} \binom{r-j}{r-i} \left\{ \binom{r-2}{r-i-1}\beta + \binom{r-2}{r-i-2} \right\}.$$

Proof. This follows immediately from (2) and Theorem 5.4. □

It is also possible to estimate b_i in terms of $n-r$. For positive integers i and x define

$$\phi_i(x) = \binom{x-2}{i-1}\binom{x-1}{0} + \binom{x-2}{i-2}\binom{x}{1} + \cdots + \binom{x-2}{0}\binom{x+i-2}{i-1}.$$

Theorem 5.7. *Suppose M is connected. Then,*

$$b_i(M) \le \phi_i(n-r)b_1(M) + \phi_{i-1}(n-r). \tag{9}$$

Proof. We can assume that every series class of M has exactly i elements. Indeed, by [a] and [c] of Theorem 3.3, any series class with more than i elements can be contracted down to cardinality i without changing either side of (9), while expanding any class with fewer than i elements may increase the left–hand side of (9) but will not alter the right–hand side. Let $\tilde{M}$ be the matroid obtained from M by contracting all of the series classes down to one element. The dual of the formula on the top of page 185 of [11] is

$$T(M;x,0) = (x^{i-1} + \dots x + 1)^{n-r} T(\tilde{M}; x^i, \frac{x^{i-1} + \dots x}{x^{i-1} + \dots x + 1}) \tag{10}$$

Using (10), we see that,

$$b_i(M) = \sum_{j=1}^{i} \binom{n-r+i-j-1}{i-j} b_j^\star(\tilde{M}). \tag{11}$$

Since $b_1^\star(\tilde{M}) = b_1(M)$, (9) follows from (11) by applying (7) to $\tilde{M}^\star$.

□

Inequality (9) is as optimal as can be expected in the sense that given $n-r, i$ and b_1 there are matroids which satisfy equality. Take any matroid which satisfies equality in (7) and expand every series class to cardinality i. Then, equality in (9) holds. Of course, since $b_r = 1$ and ϕ_i is increasing in i, no matroid can satisfy equality in (9) for all i.

6. Independence complexes

Suppose the smallest cocircuit of M has cardinality k. As pointed out in Section 4, $\Delta(M)$ is a k–CM complex. So, we can apply those methods to $\Delta(M)$. In addition to the previously mentioned absolute upper bound $h_i(M) \le \binom{n-r+i-1}{i}$ and relative upper bound $h_{i+1} \le h_i^{<i>}$, the h–vectors of independence complexes of matroids satisfy an analogue of the g–theorem for simplicial polytopes.

Theorem 6.1. [25] *Assume that M has no coloops. Let $g_i(M) = h_i(M) - h_{i-1}(M)$. Then for all $i, 1 \le i \le (r+1)/2$,*

$$g_{i+1}(M) \le g_i^{<i>}(M).$$

The above theorem was proved independently by Hausel and Sturmfels for matroids representable over the rationals using toric hyperkäler varieties [15].

Relative lower bounds, also reminiscent of the g–threorem for simplicial polytopes, were originally established by Chari using a PS–ear decomposition of $\Delta(M)$. See [14] for the definition of PS–ear decompositions and a proof of the following theorem.

Theorem 6.2. *Suppose M has no coloops. Then for all $i, 0 \le i \le r/2$.*

$$h_{i-1} \le h_i,$$

$$h_i \le h_{r-i}.$$

Problem 6.3. *Do 2–CM complexes satisfy the inequalities in the previous two theorems?*

An affirmative answer to this question would, with the addition of the Dehn–Sommerville equations, give a complete description of all possible h–vectors of simplicial homology spheres [24, Conjecture II.6.2].

In [5] Brown and Colbourn conjectured that for co-graphic M, the complex zeros of $T(M; x, 1)$ were contained in the closed unit disk. While this has since proven to be false [22], attempts to prove it led to a couple of relative lower bounds for h–vectors of independence complexes of any matroid.

Theorem 6.4. *Suppose M has no coloops.*

(1) [5] *For all $i \le r$,*

$$h_i \ge \sum_{j=1}^{i} (-1)^{j-1} h_{i-j}$$

(2) [26] *Let I_j be the number of independent subsets of M of cardinality j. Then for all $0 \le k \le r$,*

$$\sum_{j=k}^{r} \binom{j}{k} (-2)^{r-j} I_j \ge 0.$$

Stanley used the notion of a level ring to establish the relative lower bound $h_{j-i}(M) \le h_i(M) h_j(M)$ whenever $0 \le i, j \le r$. In particular, setting $j = r$, we find that $h_{r-i}(M) \le \binom{n-r+i-1}{i} h_r(M)$. By applying (6) we can obtain similar relative lower bounds for $h_{i-j}(M)$ in terms of $h_i(M)$ and we can also determine when equality occurs.

Proposition 6.5. *Assume that M has no coloops. Then for all $i, 1 \leq j < i \leq r$,*

$$h_{i-j}(M) \leq \frac{\binom{n-i+j-1}{r-i+j}}{\binom{n-i-1}{r-i}} h_i(M). \tag{12}$$

Furthermore, equality occurs if and only if every series class of M has cardinality greater than $r - i + j$.

Proof. Since M has no coloops, $\Delta(M)$ is a 2–CM complex. Therefore, (6) implies $(r - i + 1)h_{i-1}(M) \leq (n - i)h_i(M)$. In order for equality to occur, $h_i(M - e)$ must be zero for every e in E. By Theorem 3.3 [a], this is equivalent to every series class of M having cardinality greater than $r - i + 1$. The proposition follows by induction on j. □

In [6] Brown and Colbourn proved the relative lower bound $h_{r-1}(M) \leq rh_r(M)$ which only involves the rank of M. This can be improved using Theorem 5.4.

Theorem 6.6. *Let M be a rank r matroid without coloops. Then,*

$$h_{r-i} \leq \binom{r-1}{i} h_r + \binom{r-1}{i-1}. \tag{13}$$

Proof. By [9], $h_i(M)$ equals b_{r-i+1} of the free coextension of M. Since the latter matroid has rank $r + 1$ and is connected, (13) is an immediate consequence of Theorem 5.4. □

As in the case of Theorem 5.4, if $h_{r-i}(M) = \binom{r-1}{i}h_r(M)+\binom{r-1}{i-1}$, then $h_{r-j}(M) \leq \binom{r-1}{j}h_r(M) + \binom{r-1}{j-1}$ for all $0 \leq j \leq i$. A routine deletion–contraction induction shows that for a given r and h_r,

$$M = U_{1,h_r+1} \oplus \underbrace{U_{1,2} \oplus \cdots \oplus U_{1,2}}_{r-1}$$

satisfies equality in (13).

Corollary 6.7. *Let M be a rank r matroid without coloops. Let I_j be the number of independent subsets of M of cardinality j. Then,*

$$I_j \leq \sum_{i=0}^{j} \binom{r-i}{r-j} \left\{ \binom{r-1}{i} h_r + \binom{r-1}{i-1} \right\}.$$

Proof. Apply the above theorem to (1). □

In section 4 we posed the problem of finding absolute lower bounds for a k–CM complex given n and r. Here we examine this problem for independence complexes. Consider the special case of a rank two matroid M without loops. The simplification of M is isomorphic to $U_{2,m}$ where m is the number of parallel classes of M. Therefore, M is specified up to isomorphism by a partition $n = p_1 + \cdots + p_m$, where the p_i's are the sizes of the parallel classes of M. Since $h_0 = 1$ and $h_1 = n - r$, minimizing the h–vector of M is equivalent to minimizing the number of bases of M. As noted earlier, M is k–CM if and only if every hyperplane of M has cardinality at most $n - k$. Equivalently, each $p_i \leq n - k$. The number of bases of M is

$$\binom{n}{2} - \sum_{i=1}^{m} \binom{p_i}{2}.$$

This is minimized by setting $m = \lceil n/(n-k) \rceil, p_i = n-k$ for $i \leq m-1$, and $p_m = n-(m-1)(n-k)$. Note that this implies that when $n \geq 2k, h_2(M)$ is bounded below by $h_2(U_{1,n-k} \oplus U_{1,k})$.

An independence complexes is 2–CM if and only if it has no coloops. In [3] Björner showed that for any matroid without coloops $h_i \geq n-r$ for $0 < i < r$. While it is not specifically stated, the proof implies that $h_r \geq n-2r+1$. In general, given n and r there may be no single coloop-free matroid that achieves all of these bounds. For example, if $n=8$ and $r=4$, then the only matroid without coloops such that $h_4(M)=1$ is $M = U_{1,2} \oplus U_{1,2} \oplus U_{1,2} \oplus U_{1,2}$. However, $h_2(M) = 6 > n-r$. If we restrict our attention to $i < r$, then $U_{1,n-r} \oplus U_{r-1,r}$ does satisfy $h_i = n-r$ for $0 < i < r$.

Definition 6.8. $M(r,n,k) = U_{1,n-r-k+2} \oplus U_{r-1,r+k-2}$

Direct computation shows that $h_i(M(r,n,k)) = \binom{k+i-2}{i} + (n-r-k+1)\binom{k+i-3}{i-1}$. In addition, $\Delta(M(r,n,k))$ is k–CM as long as $n \geq r+2k-2$.

Theorem 6.9. *Fix $r \geq 2$ and $k \geq 3$. There exists $N(k,r)$ such that if M is a matroid without loops whose smallest cocircuit has cardinality at least k and $n \geq N(k,r)$, then for all $i, 0 \leq i \leq r$,*

$$h_i(M) \geq h_i(M(r,n,k)). \tag{14}$$

Proof. First we show that if $n > k(r+1)$, then there exists $e \in M$ such that $\Delta(M-e)$ is still k–CM. Let $\mathbf{H}$ be the set of hyperplanes of M of cardinality $n-k$. If $\mathbf{H}$ is empty, then any e will do since no hyperplane of $M-e$ will have size greater than $n-k-1$. Otherwise, let B be the intersection of all of the hyperplanes in $\mathbf{H}$. Since B is a flat of M there exists $H_1, \dots, H_{r+1}$, not necessarily distinct, in $\mathbf{H}$ such that $H_1 \cap \dots \cap H_{r+1} = B$. Therefore, $|B| \geq n-k(r+1)$ and B is not empty. But, for any $e \in B, \Delta(M-e)$ is k–CM.

As noted above, when $r=2, N(2,k)=2k$ works. So, assume that $r \geq 3$. Let M' be a contraction of M and let $n' = |E(M')|$. By Proposition 4.4, $h_i(M') \geq h_i(U_{r-1,r+k-2})$. In fact, if $n > r+k-2$, then $h_i(M')$ is strictly greater than $h_i(U_{r-1,r+k-2})$ for $1 \leq i \leq r$. Indeed, this is proved by Tutte recursion as in Proposition 4.4. The base case compares the h–vectors of $U_{2,4}$ and any five element rank two matroid whose smallest cocircuit has at least three elements. The h–vector of $U_{2,4}$ is $(1,2,3)$. From the discussion of rank two matroids, the h–vectors of the latter group of matroids is bounded below by $(1,3,4)$, the h–vector of the matroid whose simplification is $U_{2,3}$ and whose parallel classes have cardinality $2,2$ and 1. Note that this claim is not true when $k=2$. In particular, $U_{1,2} \oplus U_{r-1,r}$ is a rank r matroid without coloops and $r+2$ elements whose h_r is not strictly less than h_r of $U_{r,r+1}$.

To finish the proof, we find $N(r,k,i)$ such that the theorem holds for just h_i and then let $N(r,k)$ be the maximum of the all of the $N(r,k,i)$. Since $h_0(M)=1$ and $h_1(M) = n-r, r+k-1$ works for $N(r,k,0)$ and $N(r,k,1)$. So fix $i \geq 2$. Let N be the minimum of $h_i(\bar{M})$ for all loopless matroids $\bar{M}$ such that $|E(\bar{M})| = k(r+1)+1, r(\bar{M}) = r$ and the smallest cocircuit of $\bar{M}$ has at least k elements. Let $N(r,k,i) = k(r+1)+1+h_i(M(r,k(r+1)+1,k)) - N$.

Claim: If $n \geq N(k,r,i)$, then $h_i(M) \geq h_i(M(r,n,k))$.

Proof of claim: Choose $e_1 \in M$ such that the smallest cocircuit of $M - e_1$ has cardinality greater than or equal to k. Given e_j choose e_{j+1} so that the smallest cocircuit of $M - \{e_1, \ldots, e_j, e_{j+1}\}$ has size at least k. This can be done up to $j = n - k(r+1) - 1$. Deleting and contracting on each deletion,

$$h_i(M) = h_i(\tilde{M}) + \sum_j h_{i-1}(M - \{e_1, \ldots, e_{j-1}\}/e_j),$$

where $\tilde{M}$ is $M - \{e_1, \ldots, e_{n-k(r+1)-1}\}$. By construction, $|E(\tilde{M})| = k(r+1) + 1, r(\tilde{M}) = r$ and the smallest cocircuit of $\tilde{M}$ has at least k elements. In addition, the rank of each contraction is $r - 1$ and its independence complex is k–CM. There are two possibilities.

- Every contraction has more than $r + k - 2$ non-loop elements. In this case $h_i(M) \geq h_i(\tilde{M}) + (n - k(r+1) - 1)[h_{i-1}(U_{r-1,r+k-2}) + 1]$. Compare this to computing $h_i(M(r, n, k))$ by deleting and contracting down to $U_{1,rk-k+3} \oplus U_{r-1,r+k-2}$. The definition of $N(r, k, i)$ insures that $h_i(M)$ is bounded below by $h_i(M(r, n, k))$.
- At least one contraction, say $M - \{e_1, \ldots, e_{j-1}\}/e_j)$ has exactly $r + k - 2$ elements. Since this contraction is a rank $r - 1$ matroid whose smallest cocircuit has at least k elements it must be equal to $U_{r-1,r+k-2}$. Therefore, $M - \{e_1, \ldots, e_{j-1}\}$ has one non–trivial parallel class which contains e_j and the simplification of $M - \{e_1, \ldots, e_{j-1}\}$ is a one–element coextension of $U_{r-1,r+k-2}$. The one–element coextension of $U_{r-1,r+k-2}$ which minimizes $h_i(M - \{e_1, \ldots, e_{j-1}\})$ is the one obtained by adding a coloop to $U_{r-1,r+k-2}$. Hence, $h_i(M - \{e_1, \ldots, e_{j-1}\})$ is bounded below by $h_i(M(r, n-j, k))$. However, this implies that $h_i(M) \geq h_i(M(r, n-r, k) + j\ h_{i-1}(U_{r-1,r+k-2}) = h_i(M(r, n, k))$.

□

Some lower bound on n is necessary in order for (14) to hold. For instance, let $M = U_{1,3} \oplus U_{1,3} \oplus U_{1,3}$. Then $r = 3, k = 3$ and $n = 9$. The h–vector of M is $(1, 6, 12, 8)$, while the h–vector of $M(3, 9, 3) = U_{1,5} \oplus U_{2,4}$ is $(1, 6, 11, 12)$.
As usual, absolute lower bounds yield relative lower bounds via (6).

Corollary 6.10. *Fix $r \geq 2$ and $k \geq 3$. There exists $N(k, r)$ such that if M is a matroid without loops whose smallest cocircuit has cardinality k and $n \geq N(k, r)$, then for all $i, 0 \leq i \leq r$,*

$$(r - i + 1)h_{i-1}(M) + n\ h_i(M(r, n-1, k-1)) \leq (n - i)h_i(M).$$

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